1.1, 1.2 Notes

<u>1.1: Mathematics and Problem Solving</u>

Four-Step Problem-Solving Process:
1. Understand the problem.
   a. What is the problem giving me?
   b. What is the problem asking me?
2. Devise a plan. Some strategies include:
   a. Look for a pattern.
   b. Make a table.
   c. Draw a picture.
3. Carry out the plan.
   a. Follow your plan.
   b. Check each step as you go.
4. Look back.
   a. Check your answer.
   b. Did you answer the original question?
   c. Was there another method?
   d. Are there related problems for which you could use the same techniques?

Activity: Work with a partner to form a "book" in the following manner:
1. Get 2 slips of paper from the table.
2. Fold each piece of paper in half hamburger style, then put them together to look like a book.
3. Close your book up, and number the outside page 1, flip and number page 2, page 3, etc.

When you are finished, answer the following questions.
1. What is the total of the page numbers on one side of a sheet?

2. What is the total of the page numbers on one whole sheet?

3. What is the total of all the page numbers?

Repeat this for 3 slips of paper.

Answer the following questions.

4. Predict the sum of the page numbers if we use 4 slips of paper.

5. Without using a calculator, determine the value of 1 + 2 + ... + 24.

6. Write a formula for the sum of the page numbers with *n* sheets.

Let's look at the actual problem solving approach:
Example: If you create a book out of *n* sheets, what is the sum of the page numbers?

Step 1 - Understand the Problem: What do we need to understand?

Step 2 - Devise a Plan: Our strategy was ___________________________.
How did we do it?

Step 3 - Carry Out the Plan: Using the 2 sheet and 3 sheet example, we figured out how to determine the last page number and how to find (1) the total page numbers on each side of a sheet, (2) the total page numbers on each sheet, and (3) the sum of all page numbers. We then used this information to find the formula for the sum of the pages of a book made from *n* sheets.

Step 4 - Look Back:
We can check our answers for the 2 case: 1+2+3+4+5+6+7+8=36.
Did we answer the question?

Let's try a different approach.

Example: What is the sum of the first *n* positive integers?

Strategy: Examine a Related Problem

Example: What is the sum of the even numbers less than or equal to 40? Have we done a similar problem before?

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Strategy: Examine a Simpler Case

Example: If there are 10 people in the room and each person shakes every other person's hand, how many handshakes were performed? Nobody shakes hands twice, and shaking your own hand would make you look weird...

Strategy: Identify a sub-goal. In some problems, you may want to find a piece of information that will help you solve the problem first.

Strategy: Make a diagram.

Example: To get some cardio exercise, you climb the stairs in a tall building. You start from the first floor (Floor 1). You then go up 3 floors, down 2 floors, up 7 floors, down 5 floors, and then up 7 floors to stop at the top floor. How many floors does the building have?

Strategy: Use Indirect Reasoning. Sometimes it is easier to show that the opposite can't happen. An important form of this is the process of elimination.

Example: Andrew, Michael, and Travis played a strategy game. Michael did not come in first place, as usual. Travis beat Michael, but he did not come in first place. Who took first, second, and third place?

Strategy: Make a Table

Example: A wealthy family hired a maid and a gardener. The maid comes in every 2 days, and the gardener comes in every 3 days. If they started on the same day, how many days will go by before they come in on the same day again?

Strategy: Guess and Check

Example: Find two numbers whose product is 42 and whose sum is 17.

Strategy: Work Backward

Example: You have an 80 average on 6 quizzes. Your teacher tells you that you can drop your lowest quiz grade of 30. What is your new average?

Strategy: Use Direct Reasoning.

Example: If two people won 3 games of checkers each, what is the minimum number of games played?

Strategy: Write an Equation. This was discussed heavily in Math 111.

<u>1.2: Patterns</u>

What appears to be a pattern may need further checking. You need enough data to identify a pattern.

Example: Fill in the following pattern: 13, 19, 25, ____, ____, ____

Example: Fill in 2, 4, ____, ____, ____ in as many ways as you can think of.

2.1 Notes

2.1: Base 10 and Base 5 Numeration Systems

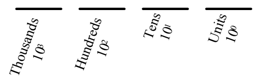
Definition: If a is any number and n is any natural number, then

$a^n = a \times a \times a \times \ldots \times a$ (n factors)

Our number system is called the Hindu-Arabic numeration system, and it is a base 10 number system using the characters 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. (Note that there are 10 characters.)

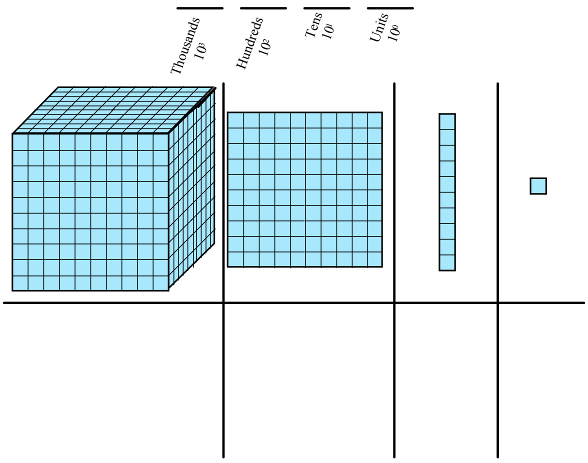
What does this mean? When a number is written in base 10, each "place value" corresponds to a power of 10.

Example: The number 6143 means "6 thousands, 1 hundred, 4 tens, and 3 ones".



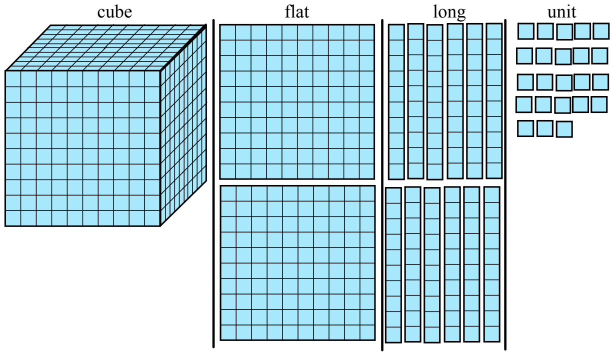
Another perspective: We can also write the number 6143 in expanded form as $6143 = 6 \cdot 10^3 + 1 \cdot 10^2 + 4 \cdot 10^1 + 3 \cdot 10^0$

Example: Represent the number three hundred five in base 10.



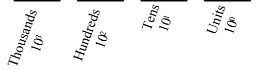
Example: If you have 1 cube, 2 flats, 12 longs, and 23 units, what is the minimum number of blocks you can have using a fair trade?

10 units = 1 long   10 longs = 1 flat   10 flats = 1 cube

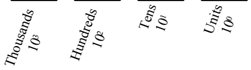


Example: If you have 1 cube, 2 flats, 12 longs, and 23 units, what is the minimum number of blocks you can have using a fair trade?

Consider filling the diagram below in the same manner. Is this number valid?



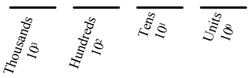
We showed that this number is the same as this one:

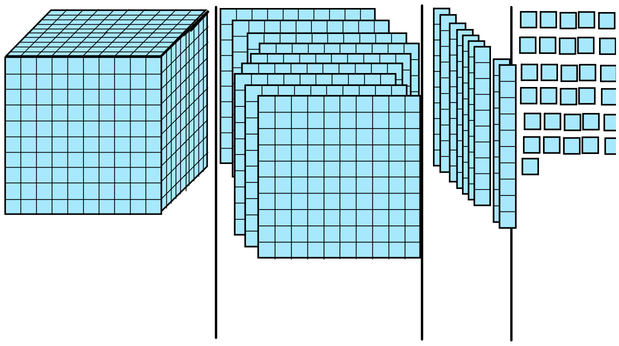


This gives us an important fact about the base 10 number system. You cannot have more than 9 in a single "place value".

Example: If you have 9 flats, 9 longs, and 31 units representing a base 10 number, perform the necessary exchanges to write it in the proper form.

10 units = 1 long
10 longs = 1 flat
10 flats = 1 cube





2.1 Notes

Definition: The base 5 number system uses the characters 0, 1, 2, 3, and 4 and each "place value" corresponds to a power of 5.

Notation: We denote a number in base five by writing "five" (preferred) or "5" in a subscript.

Example: The number 2143_{five} means "2 5^3's, 1 5^2, 4 5^1's, and 3 ones".

___ ___ ___ ___
5^3 5^2 5^1 Units 5^0

Let's count the first 30 base 5 numbers:

Example: The number 2143_{five} means "2 5^3's, 1 5^2, 4 5^1's, and 3 ones".

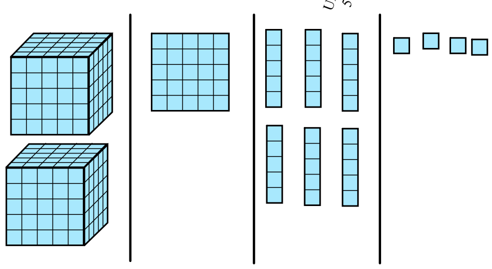
___ ___ ___ ___
5^3 5^2 5^1 Units 5^0

What does this number mean in base 10? Let's try expanded form.

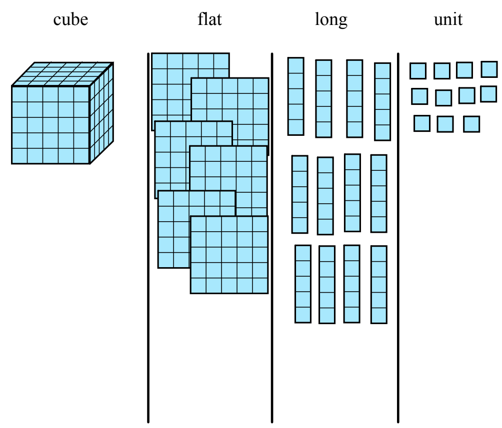
Note: A number without a base written is assumed to be base ten.

Example: What is wrong with this picture?
General Rule:

2 1 6 4
5^3 5^2 5^1 Units 5^0



Example: If you have 1 cube, 6 flats, 12 longs, and 11 units, what is the minimum number of blocks you can have using a fair trade?



Example: If you have 1 cube, 6 flats, 12 longs, and 11 units, what is the minimum number of blocks you can have using a fair trade?

We showed that this description gives us the following base 5 number:

___ ___ ___ ___
5^3 5^2 5^1 Units 5^0

What is this number in base 10?

Conversions: One method to convert a number from base 10 to base 5 uses a form of repeated long division.

Example: Convert 423_{ten} to base 5.

___ ___ ___ ___
5^3 5^2 5^1 Units 5^0

2.1 Notes

Example: Convert 149_{ten} to base 5.

Example: Convert 575_{ten} to base 5.

Example: Convert 423_{ten} to base 5. (This was the first example.)

Different Method:

3143_{five}

Bonus for a free quiz:

Write up an explanation for why this works and turn it in tomorrow. If someone explains why it works to the class, all of you may use it.

2.2 Notes

2.2: Describing Sets

Definition: A set is any collection of objects with no repetitions. An object in a set is said to be an element of the set. One way to write a set is to list them in { } with commas in between the elements.

Notation: If A is a set and a is an element of A, we write $a \in A$. If b is not an element of A, we write $b \notin A$.

Example: Write the set of the first five counting numbers and give examples of elements in and not in the set.

Definition: (Set builder notation) Let S be a set. Then we can write $S = \{x \mid x \text{ satisfies some conditions}\}$. This is read "$S$ equals the set of elements x such that x satisfies some conditions".

Another way to think of set builder notation is {form of elements | conditions}. This will show up more in the examples.

Example: Write $S = \{1, 2, 3, 4, 5\}$ in set builder notation.

Definition (Special Sets):

(1) The Natural Numbers: $\mathbb{N} = \{1, 2, 3, 4, ...\}$
(2) The Integers: $\mathbb{Z} = \{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$
(3) The Real Numbers: $\mathbb{R} = \{x \mid x \text{ is any number that can be written as a decimal}\}$

Example: Use set builder notation to write the set of all real numbers between 0 and 1.

Example: Use set builder notation to write the set of all even integers.

Example: Use set builder notation to write the set of perfect squares 1, 4, 9, 16, etc.

Example: Describe the elements of the following sets.

(a) $\{3x \mid x \in \mathbb{Z}\}$

(b) $\{-x \mid x \in \mathbb{N}\}$

(c) $\{a/b \mid a, b \in \mathbb{Z}, b \neq 0\}$

Definition: Two sets are equal if they contain exactly the same elements in any order.

Definition: The cardinal number of a set S, denoted $n(S)$ or $|S|$, is the number of elements of S.

Definition: The empty set, denoted $\varnothing$, is the set with no elements. The empty set can also be written as { }.

Definition: A set is a finite set if the cardinal number of the set is 0 or a natural number. A set with infinitely many elements, such as the natural numbers, is called an infinite set.

Example: Find the cardinal number of $A = \{1, 2, 3, 4\}$, $B = \{0\}$, $C = \{2, 4, 6, 8, ...\}$, and $\varnothing$.

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Example: Find the cardinal number of the following sets.

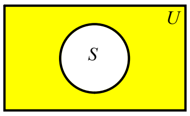
(a) $S = \{1, 4, 7, 10, 13, ..., 40\}$

(b) $T = \{33, 37, 41, 45, 49, ..., 353\}$

Definition: The universal set, denoted U, is the set of all elements being considered in a given discussion.

Definition: The complement of a set S, denoted $\overline{S}$, is the set of all elements in U that are not in S. That is, $\overline{S} = \{x \mid x \in U \text{ and } x \notin S\}$.

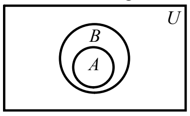
A complement can be thought of in the following manner. The shaded region is $\overline{S}$:



Example: If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, find the complements of $A = \{2, 4, 6, 8\}$ and $B = \{1, 2, 4, 6, 7\}$.

Definition: If A and B are sets, we say that A is a subset of B, denoted $A \subseteq B$, if every element of A is an element of B. If $A \subseteq B$ and $A \neq B$, we say that A is a proper subset of B, denoted $A \subset B$.

A subset can be thought of in the following manner. In the figure $A \subseteq B$:



Example: Fill in the blanks with either $\subseteq$ or $\not\subseteq$.

$\{1, 2, 3, 4\}$ ___ $\{1, 2, 3, 4, 5\}$	$\{1, 2, 3, 4\}$ ___ $\{1, 2, 3, 4\}$
$\{1, 2, 3, 4, 5\}$ ___ $\{1, 2, 3, 4\}$	$\{1, 2, 3, 4, 5\}$ ___ $\{1, 2, 3, 4, 6\}$
$\{0\}$ ___ $\{1, 2, 3, 4\}$	$\varnothing$ ___ $\{1, 2, 3, 4\}$
$\{1, 2, 3, 4\}$ ___ $\varnothing$	$\varnothing$ ___ $\varnothing$

Example: Fill in the blanks with either $\in$, $\notin$, $\subseteq$, or $\not\subseteq$.

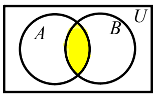
$\{2\}$ ___ $\{1, 2, 3\}$	0 ___ $\mathbb{N}$
2 ___ $\{1, 2, 3\}$	$\mathbb{Z}$ ___ $\mathbb{N}$
5 ___ $\{1, 2, 3, 4\}$	5 ___ $\{2x \mid x \in \mathbb{Z}\}$
$\varnothing$ ___ $\{1\}$	$\mathbb{R}$ ___ $\mathbb{R}$
0 ___ $\varnothing$	$\{a/b \mid a, b \in \mathbb{Z}, b \neq 0\}$ ___ $\mathbb{R}$
$\{4\}$ ___ $\{2\}$	$\{1.5\}$ ___ $\mathbb{N}$

2.3 Notes

2.3: Other Set Operations

Definition: If A and B are sets, the intersection of A and B, denoted $A \cap B$, is the set of elements that are in **both** A **and** B. That is,
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

An intersection can be thought of in the following manner. The shaded region is $A \cap B$:



Examples:

$\{1, 2, 3, 4\} \cap \{2, 4, 6, 8\} =$

$\{x^2 \mid x \in \mathbb{Z}\} \cap \{1, 2, ..., 20\} =$

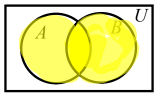
$\{1, 2\} \cap \{1, 2, 3\} =$

$\{1, 2\} \cap \{3, 4\} =$

$\varnothing \cap \{1,2\} =$

Definition: If A and B are sets, the union of A and B, denoted $A \cup B$, is the set of elements that are in **either** A **or** B. That is,
$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

A union can be thought of in the following manner. The shaded region is $A \cup B$:



Examples:

$\{1, 2, 3, 4\} \cup \{2, 4, 6, 8\} =$

$\{x^2 \mid x \in \mathbb{Z}\} \cup \{1, 2, ..., 20\} =$

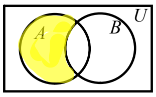
$\{1, 2\} \cup \{1, 2, 3\} =$

$\{1, 2\} \cup \{3, 4\} =$

$\varnothing \cup \{1,2\} =$

Definition: If A and B are sets, the set difference of B and A (or relative complement of A relative to B), denoted $A - B$ and read "A set minus B", is the set of elements that are in A but not in B. That is,
$A - B = \{x \mid x \in A \text{ and } x \notin B\}$.

A set difference can be thought of in the following manner. The shaded region is $A - B$:



Examples:

$\{1, 2, 3, 4\} - \{2, 4, 6, 8\} =$

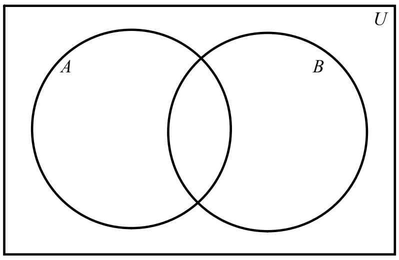
$\{1, 2, ..., 20\} - \{x^2 \mid x \in \mathbb{Z}\} =$

$\{1, 2\} - \{1, 2, 3\} =$

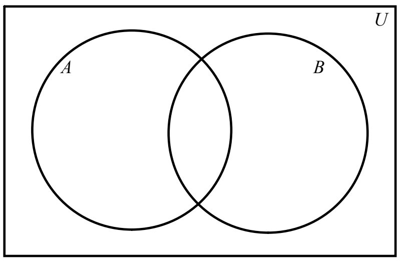
$\{1, 2\} - \{3, 4\} =$

$\varnothing - \{1,2\} =$

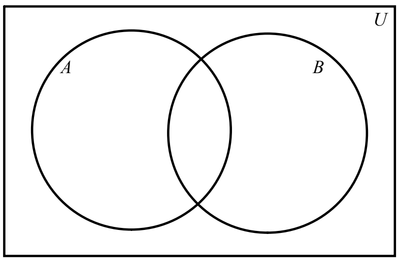
Example: Draw the Venn Diagram for $A \cup \overline{B}$.



Example: Draw the Venn Diagram for $\overline{A} \cup \overline{B}$.

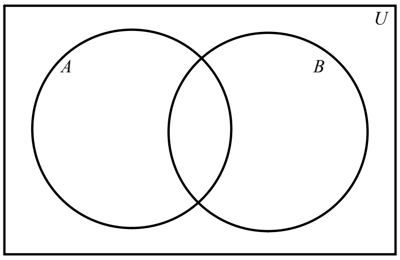


Example: Draw the Venn Diagram for $A \cap \overline{B}$.

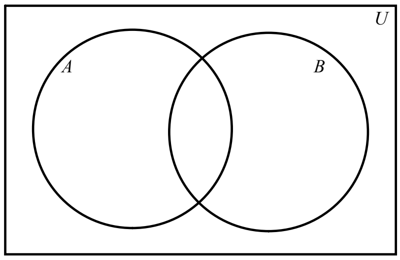


2.3 Notes

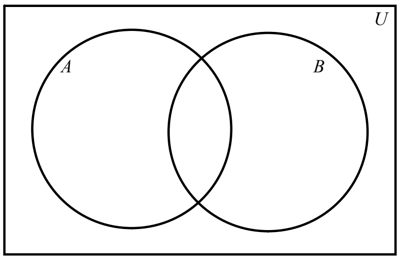
Example: Draw the Venn Diagram for $\overline{A} \cap \overline{B}$.



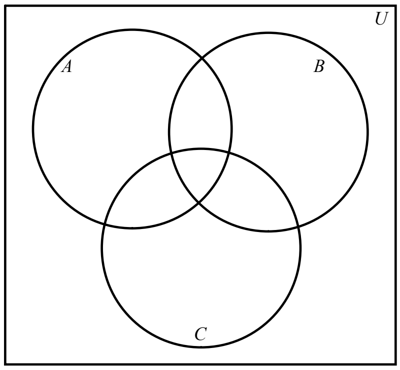
Example: Draw the Venn Diagram for $\overline{A} - B$.



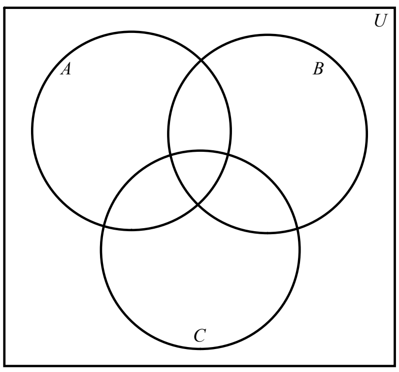
Example: Draw the Venn Diagram for $\overline{A - B}$.



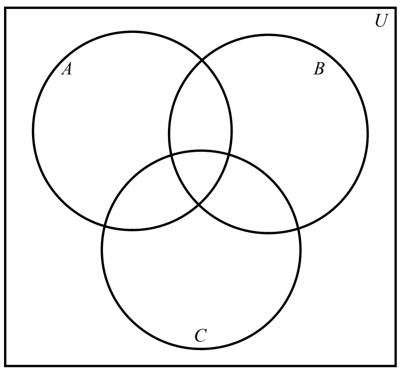
Example: Draw the Venn Diagram for $(A \cap B) \cup C$.



Example: Draw the Venn Diagram for $(A \cup B) - C$.



Example: Draw the Venn Diagram for $A \cap (\overline{B \cup C})$.



3.1 Notes

3.1: Addition and Subtraction of Whole Numbers

Definition: The whole numbers, denoted W, are the natural numbers and 0. In other words, $W = \{0, 1, 2, 3, ...\}$.

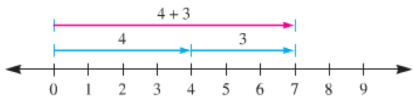
Definition: (Addition of Whole Numbers) Let A and B be two disjoint (no common elements) finite sets. If $n(A) = a$ and $n(B) = b$, then $a + b = n(A \cup B)$.

Definition: The numbers a and b are called the addends and $a + b$ is called the sum.

Addition Model 1: Representing the sum of two numbers by combining two sets is known as the set model of addition

Example: You have 3 cats and 4 dogs. How many animals do you have total?

Addition Model 2: The number line model is another way to represent addition, in which we draw a number line and represent numbers by arrows pointing right whose length is the same as the number. When we add numbers, we place the second number's line at the tip of the first number's line, and then we look at the whole length.



Example: If you were on the fourth floor of a building and moved up 3 floors, what floor are you now on?

Example: Determine which models you would use for each situation.

Your cat weighed 8 pounds and gained 2 pounds last month. How much does he weigh now?

You go strawberry picking and you have 6 strawberries. If you pick 5 more strawberries, how many do you have?

You have 1 laptop and 2 desktop computers in your house. How many computers do you have total?

Definition: For any whole numbers a and b:

1. We say that a is less than b, denoted $a < b$, if and only if there exists a natural number k such that $a + k = b$.
2. We say that a is less than or equal to b, denoted $a \leq b$, if and only if $a < b$ or $a = b$.
3. We say that a is greater than b, denoted $a > b$, if and only if $b < a$.
4. We say that a is greater than or equal to b, denoted $a \geq b$, if and only if $b \leq a$.

Examples:

Definition: For any whole numbers a and b such that $a \geq b$, $a - b$ is the unique whole number c such that $b + c = a$.

Subtraction Model 1: The Take-Away Model is used when you start with a set and remove elements. (Note: This idea could be used to define subtraction in terms of subsets and set difference.)

Example: You have 9 apples and you give 4 away to your teachers. How many apples do you have left?

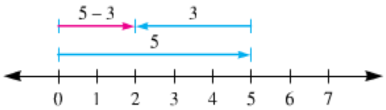
Subtraction Model 2: The Missing Addend Model is used when you are determining how much is left to get to a certain number.

Example: You have 3 dollars and you want to buy a book that costs 7 dollars. How much more money do you need to buy the book?

Subtraction Model 3: The Comparison Model is used when you have two sets and you want to find how many more elements are in one set than another.

Example: You ate 2 pieces of cake and your friend ate 5 pieces of cake. How many more pieces of cake did your friend eat than you did?

Subtraction Model 4: The Number-Line Model is used when you are given a sum and the number that was added to get there, and you want to find out what the original number was.



Example: You drove 100 miles in a two hour period. If you drove 40 miles in the second hour, how many miles did you drive during the first?

Example: Determine which models you would use for each situation.

Michael is driving the 300 mile trip to Virginia. If he has driven 90 miles, how many more miles does he have to drive?

Susan ate 56 donuts in two months. If you ate 6 donuts in the second month, how many donuts did you eat in the first month?

You earned 100 points on the last test and the student next to you earned 90. How many more points did you earn than the student next to you?

Your cat weighed 15 pounds and lost 2 pounds last month due to a much needed diet. How much does he weigh now?

Theorem: The following properties hold for addition of whole numbers:

1. (Closure) If a and b are whole numbers, then $a + b$ is a whole number.
2. (Commutative) If a and b are whole numbers, then $a + b = b + a$.
3. (Associative) If a, b, and c are whole numbers, then $(a + b) + c = a + (b + c)$.
4. (Identity) There is a unique whole number (0 in this case), the additive identity, such that for any whole number a, $a + 0 = 0 + a = a$.

Example: Given the following sets, determine which of these properties it has.

(a) {2, 4, 6, 8, 10, ...}

(b) {0, 2, 4, 6, 8}

Example: Given the following sets, determine which of these properties it has.

(c) {1, 3, 5, 7, 9, ...}

(d) $\{x^2 \mid x \in W\}$

Question: Which of the properties of addition of whole numbers are satisfied with subtraction of whole numbers?

Closure:

Commutative:

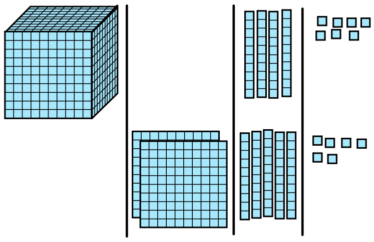
Associative:

Identity:

3.2 Notes

3.2: Algorithms for Addition and Subtraction

Example: (The Standard Algorithm) Add 1047 and 256.
Note: Nowadays, we use the word "trade" or "regroup" instead of "carry".



Standard Algorithm:

```
  1047
+  256
```

Example: (The Left to Right Algorithm)

```
  1047          2359
+  256        + 5667
```

Why does this work?

Example: (The Lattice Algorithm)

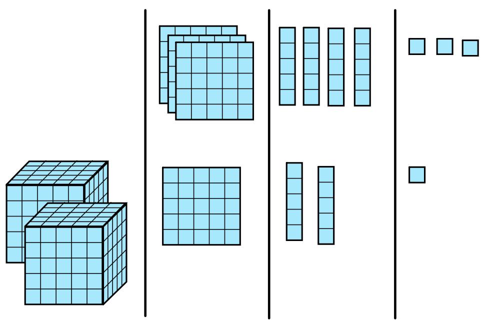
```
  1   0   4   7          1   7   9   4
+     2   5   6        + 3   2   9   7
```

```
  2   3   5   9
+ 5   6   6   7
```

Why does this work?

Base 5 Addition: We can use all of the previous algorithms with base 5 numbers using this table.

Example: Add 343_{five} and 2121_{five}.



Example: All numbers are base 5.

Standard:

```
   343
+ 2121
```

```
  2103
+ 2244
```

Left to Right:

```
   343
+ 2121
```

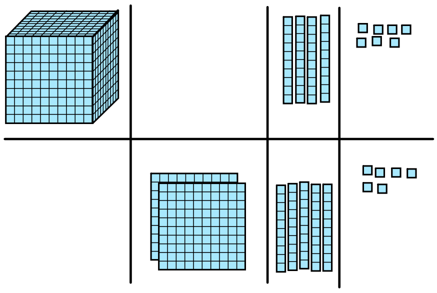
Lattice:

```
      3   4   3
+ 2   1   2   1
```

```
  2   1   0   3
+ 2   2   4   4
```

3.2 Notes

Example: (The Standard Algorithm) 1047 - 256.



Standard Algorithm:

```
  1 0 4 7
-   2 5 6
```

Example: (Equal Additions Algorithm)

```
  1047
-  256
```

```
  2359
-  467
```

```
  5238
-  478
```

Why does this work?

Example: All numbers are base 5.

Standard:

```
  1 1 3 2
-   4 3 4
```

```
  4 3 0 2
- 3 4 2 3
```

Equal Additions:

```
  1132
-  434
```

```
  4302
-3423
```

Example: All numbers are base 5.

Standard:

```
  1 0 1 2
-     1 3
```

```
  3 0 0 1
- 1 3 2 3
```

Equal Additions:

```
  1012
-   13
```

```
  3001
-1323
```

3.3 Notes

3.3: Multiplication and Division of Whole Numbers

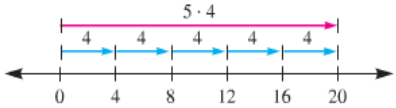
Definition: (Multiplication of Whole Numbers) For any whole numbers a and n, where $n \neq 0$, $n \times a = a + a + ... + a$ (n terms). This can also be written $n \cdot a$ or just na.

Definition: The numbers n and a are called the factors and na is called the product.

Multiplication Model 1: Representing the product of two numbers by adding numbers multiple times is known as the repeated addition model

Example: A king size candy bar costs \$1. If you have a huge craving for chocolate and buy five of these, how much do you spend?

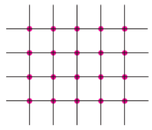
Multiplication Model 2: The number line model is another way to represent multiplication, in which we draw a number line and represent numbers by arrows pointing right whose length is the same as the second number. We place arrows for the second number one after the other, and there are as many arrows as the first number. The product is the whole length.

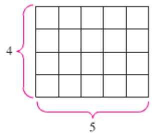


Example: If you drive down the interstate at 70 mph for 3 hours, how far did you go?

Note: The repeated addition and number line models are virtually the same thought process, so either answer is acceptable to me when one works.

Multiplication Models 3 and 4: The array model and the area model are used to represent when a situation is easily arranged into a grid with rows and columns.

Array: 

Area: 

Example: (Array Model) If you have a sheet of stickers with 15 rows of 10 stickers, how many stickers are on the sheet?

Example: (Area Model) If a sheet of paper is 10 in. by 12 in., what is the area of the sheet of paper?

Note: The array and area models are virtually the same thought process, so either answer is acceptable to me when one works.

Multiplication Model 5: The Cartesian Product Model is used to represent when you have two sets and you are looking at all combinations of elements where 1 element is from the first set and 1 element is from the second set.

Example: Tyler has 20 shirts and 8 pairs of pants. How many ways can he combine them for one outfit?

Example: Determine which models you would use for each situation.

(a) Your room is 12 ft by 14 ft. What is the area of your floor?

(b) Water is flowing out of the faucet at 300 mL per second for 30 seconds. How much water comes out of the faucet?

(c) A refrigerator magnet weighs 3 ounces. If you have 4 of these, how much do they weigh altogether?

(d) You have 10 different pairs of socks that go well with any shoe and you have 1,394 pairs of shoes. How many ways can you fashionably cover your feet?

(e) A base ten flat is actually composed of units. Each row has 10 units and each column has 10 units. How many units are in a flat?

Theorem: The following properties hold for multiplication of whole numbers:

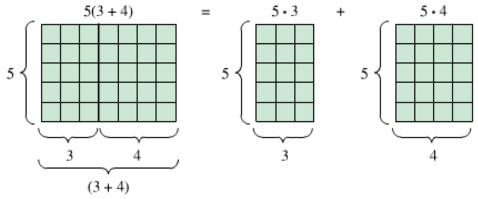
1. (Closure) If a and b are whole numbers, then $a \times b$ is a whole number.
2. (Commutative) If a and b are whole numbers, then $a \times b = b \times a$.
3. (Associative) If a, b, and c are whole numbers, then $(a \times b) \times c = a \times (b \times c)$.
4. (Identity) There is a unique whole number (1 in this case), the multiplicative identity, such that for any whole number a, $a \times 1 = 1 \times a = a$.
5. (Zero Product) For any whole number a, $a \times 0 = 0 \times a = 0$.

3.3 Notes

Theorem: The following properties hold for multiplication of whole numbers:

6. (Distributive) For any whole numbers a, b, and c, $a(b + c) = ab + ac$. Similarly, $a(b - c) = ab - ac$.

Rationale:



Definition: (Division of Whole Numbers) For any whole numbers a and b, where $b \neq 0$, $a \div b = c$ if and only if c is the unique whole number such that $b \times c = a$.

Definition: The number a is called the dividend, the number b is called the divisor, and the number c is called the quotient.

Division Model 1: The partition model is used to represent when you have a set of elements and wish to distribute them equally (partition) them into smaller groups. You then determine how many elements are in each group.

Example: You baked a birthday cake and cut it into 18 pieces. If there are 9 people eating birthday cake and they eat equal portions, how many pieces does each person get?

Division Model 2: The measurement model is used to represent when you have a set of elements and wish to make groups with the same amount in each group. You then determine how many groups you can make.

Example: You baked a birthday cake and cut it into 18 pieces. If you give 2 pieces to each person, how many people can you serve?

Example: Determine which models you would use for each situation.

A group of 10 people run a car wash and make a $620 profit. If they split the profits equally, how much should each person get?

You have a bucket containing 150 pieces of Halloween candy. If you give 5 pieces of candy to each trick-or-treater, how many trick-or-treaters can you give candy to?

The Division Algorithm: Given any whole numbers a and b with $b \neq 0$, there exists whole numbers q (quotient) and r (remainder) such that $a = bq + r$, where $0 \leq r < b$.

Example: If $a = 132$ and $b = 10$, what are the values of q and r?

Example: If 101 is divided by a number and the remainder is 21, what are the possible divisors?

Division with Zero: Let n be a whole number.

$0 \div n = 0$ (if $n \neq 0$) because the only c that satisfies $nc = 0$ is $c = 0$.

How about $n \div 0$? Why is it undefined?

3.4 Notes

3.4: Algorithms for Multiplication and Division

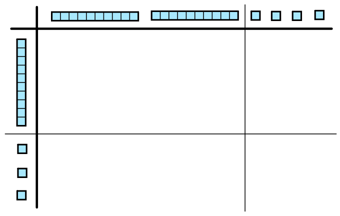
Example: (The Standard Algorithm - Single Digit)

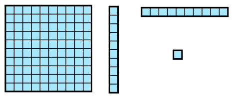
Multiply 3037 and 4.

$$\begin{array}{r} 3\;0\;3\;7 \\ 3\;0\;3\;7 \\ 3\;0\;3\;7 \\ +\;3\;0\;3\;7 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 3\;0\;3\;7 \\ \times\qquad 4 \\ \hline \end{array}$$

Example: (The Standard Algorithm - Multiple Digit)

Multiply 24 and 13.





Standard Algorithm:

$$\begin{array}{r} 2\;4 \\ \times\;1\;3 \\ \hline \end{array}$$

Example: (Lattice Multiplication)

(a) Multiply 23 and 14.

Why does this work?

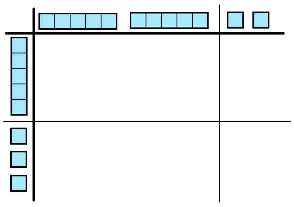
Example: (Lattice Multiplication)

(b) Multiply 75 and 20.

(c) Multiply 273 and 54.

Base 5 Multiplication: We can use both of the previous algorithms with base 5 numbers using this table.

Example: Multiply 14_{five} and 22_{five}.



Standard Algorithm:

$$\begin{array}{r} 1\;4 \\ \times\;2\;2 \\ \hline \end{array}$$

3.4 Notes

Example: All numbers are base 5.

Standard:

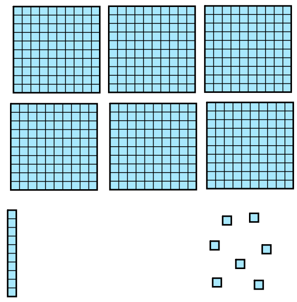
$$\begin{array}{r} 1\ 4 \\ \underline{\times\ 2\ 3} \end{array}$$

$$\begin{array}{r} 3\ 3 \\ \underline{\times\ 3\ 2} \end{array}$$

Lattice:

The Long Division Algorithm:

Example: Calculate 617 ÷ 5



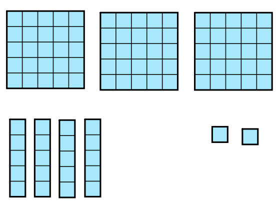
$5 \overline{)\,6\ 1\ 7}$

Example: Carefully explain why this works:

$23 \overline{)\,1\ 9\ 8\ 6}$

Example: Calculate $342_{five} \div 2_{five}$.

Let's try $342_{five} \div 2_{five}$ with Base 5 blocks.



Example: Calculate $213_{five} \div 3_{five}$.

3.4 Notes

Example: Calculate $1322_{five} \div 32_{five}$.

Example: Calculate $2002_{five} \div 21_{five}$.

4.1 Notes

4.1: Divisibility

Definition: For two whole numbers a and b, $b \neq 0$, we say b divides a, written as $b \mid a$, if $a \div b$ is a whole number. Other ways to say this are "a is divisible by b", "b is a divisor of a", "a is a multiple of b", and "b is a factor of a".

Divisibility Rules: Let n be a whole number.

2 | n if and only if n ends in an even number.

$3 \mid n$ if and only if the sum of the digits of n is divisible by 3.
Example: Show $3 \mid 5352$.

$4 \mid n$ if and only if 4 divides the last 2 digits of n.
Example: Show $4 \mid 1880$.

$5 \mid n$ if and only if n ends in either 0 or 5.

Divisibility Rules: Let n be a whole number.

$6 \mid n$ if and only if $2 \mid n$ and $3 \mid n$.

For 7, form a new number k by taking off the last digit of n and subtracting its double from the result. Then $7 \mid n$ if and only if $7 \mid k$.
Example: Show that $7 \mid 3654$.

$8 \mid n$ if and only if 8 divides the last 3 digits of n.

$9 \mid n$ if and only if the sum of the digits of n is divisible by 9.

$10 \mid n$ if and only if n ends in 0.

For 11, we form a new number k by adding then subtracting the digits of n. It is important that we consider the sign of the first digit as part of this addition and subtraction. Then $11 \mid n$ if and only if $11 \mid k$.
Example: Show that $11 \mid 1485$.

Example: Determine which of the numbers 2 through 11 divide 1680. Justify each of your tests.

Example: Determine which of the numbers 2 through 11 divide 13860. Justify each of your tests.

More Examples: These problems would be good to try before your quiz.

Determine which of the numbers 2 through 11 divide the following numbers. Justify each of your tests.

(a) 6048
(b) 3300
(c) 7777
(d) 20,790

Activity: Invent divisibility rules for as many numbers less than 30 that you can find. There may be multiple rules for the same number. Warning: 13, 17, 19, 23, 27, and 29 are difficult.

Try to find at least one general rule for certain types of numbers.

4.2 Notes

4.2: Prime and Composite Numbers

Definition: A prime number is a number with exactly two distinct positive factors, namely 1 and themselves.

Definition: A composite number is a number with more than two distinct positive factors:

Is 1 a prime number or a composite number?

Find which numbers are prime in the set {1, 2, ..., 100}

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

This is known as the Sieve of Eratosthenes.

Theorem: If n is composite, then it has a prime factor p with the property that $p^2 \leq n$.

In other words, to see if a number is prime, we need only check all of the possible prime factors up to its square root.

Proof:

Example: List the factors of 28. Is 28 prime or composite?

Is 301 prime?

Is 307 prime?

Fundamental Theorem of Arithmetic: Each composite number can be written as a product of primes in exactly one way (ignoring the order of the factors).

Definition: This product described above is known as the prime factorization of a number.

Example: What is the prime factorization of 120?

Example: What is the prime factorization of 270?

4.3 Notes

4.3: Greatest Common Divisor and Least Common Multiple

Definition: The greatest common divisor (GCD) of two natural numbers a and b is the greatest natural number that divides both a and b.

Intersection of Sets Method: We write the set of all divisors of each number and then intersect these sets to find the common divisors. The largest element of the intersection is the greatest common divisor.

Example: Find the GCD of 12 and 18.

Prime Factorization Method: We find the prime factorization of both numbers. The GCD is the product of the **common** primes, raised to the **lowest** power that shows up in either prime factorization.

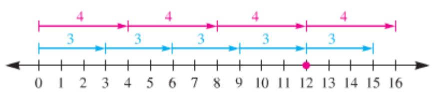
Note: If there are no common primes, then the only common factor is 1, so the GCD of these two numbers is 1. In this case, we say that the numbers are relatively prime.

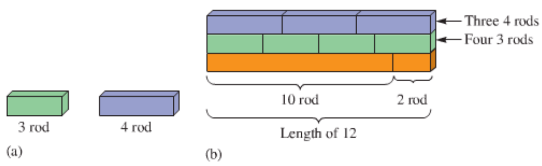
Example: Find the GCD of 192 and 360.

Example: Use the Intersection of Sets and Prime Factorization methods to find the GCD of 56 and 84.

Definition: The least common multiple (LCM) of two natural numbers a and b is the least natural number that is both a multiple of a and a multiple of b.

Not Tested: (Number-Line Method/Colored Rods Method) Find the LCM of 3 and 4.





Intersection of Sets Method: We write the set of all multiples of each number and then intersect these sets to find the common multiples. The smallest element of the intersection is the least common multiple.

Example: Find the LCM of 6 and 8.

Prime Factorization Method: We find the prime factorization of both numbers. The LCM is the product of all of the primes in **either** number, raised to the **greatest** power that shows up in either prime factorization.

Example: Find the LCM of 840 and 792.

4.3 Notes

Example: Use the Intersection of Sets and Prime Factorization methods to find the LCM of 12 and 14.

Theorem: For any two natural numbers *a* and *b*,
GCD(a, b) $\times$ LCM(a, b) = $a \times b$.

Proof: See handout on Blackboard. It will appear shortly after class.

Example of why this works: Consider the numbers 8 and 12.

$8 = 2^3 = 2^3 \times 3^0$
$12 = 2^2 \times 3^1$

By considering a 3^0, we can take all primes and look at the smallest powers for the GCD and the largest powers for the LCM.

GCD(8, 12) = $2^2 \times 3^0$
LCM(8, 12) = $2^3 \times 3^1$

So, GCD(8,12) $\times$ LCM(8,12) = $2^5 \times 3^1 = (2^3 \times 3^0) \times (2^2 \times 3^1) = 8 \times 12$.
Notice that for each prime, the GCD picks one power and the LCM picks the other power.

5.1 Notes

<u>5.1: Integers and the Operations of Addition and Subtraction</u>

Notation: To avoid confusion between subtraction and negative numbers, a negative number will be denoted with a superscript -, such as ${}^-5$. We read ${}^-x$ as "the opposite of x".

Definition: The negative integers are the <u>opposite</u> of the positive integers, defined to be the value with the same distance from 0 but on its left side.

Example: Find the opposite of the following:

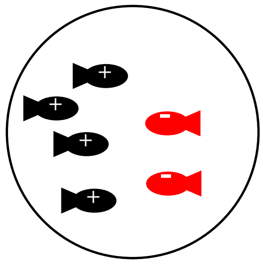
$x = 2$

$x = {}^-3$

$x = 0$

Addition of Integers:

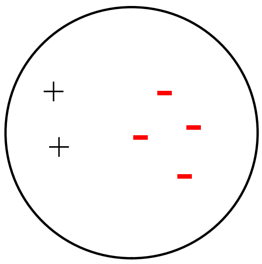
Chip Model: The chip model represents integers by using black chips for positive numbers and red chips for negative numbers. To make this more interesting, I use the concept of "posi-fish" and "nega-fish". When a posi-fish and a nega-fish are together, they slam into each other and destroy each other.



$4 + {}^-2 = 2$

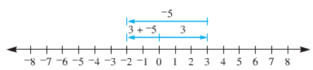
Charged Field Model: This model is virtually identical to the chip model, but we use just signs instead of images. The positive charges would "neutralize" the negative charges.

We will often use this model, but the other model is great for children.

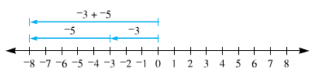


$2 + {}^-4 = {}^-2$

Number Line Model: You are standing on a number line facing the positive direction, then you move forward if adding a positive number or backward if adding a negative number.



$3 + {}^-5 = {}^-2$



${}^-3 + {}^-5 = {}^-8$

Example: Represent the problem $1 + {}^-5 = {}^-4$ using each method.

Definition: The <u>absolute value of x</u>, denoted $|x|$, is the distance from 0 to x. Since distance is always positive, the absolute value of a number is just the positive version of that number. However, to define this more carefully:

$|x| = x$ if $x \geq 0$ and $|x| = {}^-x$ if $x < 0$

Special Rules for Addition of Integers: The following set of rules is not in the text, but it can be taught to students who are struggling to speed up their calculations. This is NOT, however, a substitute for these models, as students should have a deep understanding of whether the solution is positive or negative before learning these.

1. If you are adding two numbers with the same sign, add their absolute values and keep the common sign.
2. If you are adding two numbers with different signs, subtract the number with the smaller absolute value from the number with the larger absolute value, then use the sign of the number with the larger absolute value.

5.1 Notes

Properties of Integer Addition:

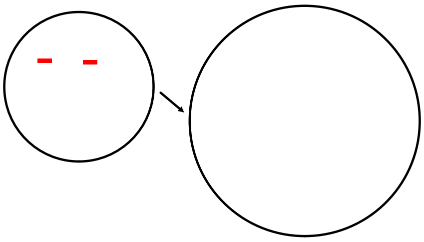
Closure, Commutative, Associative, and Identity still hold.

5. Additive Inverse Property: For every integer a, there is a unique integer ^-a such that $a + {}^-a = {}^-a + a = 0$.

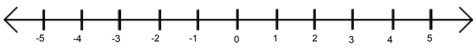
Other Properties:
6. $^-(^-a) = a$
7. $^-(a + b) = {}^-a + {}^-b$

Charged Field Model: We again just use signs instead of images.

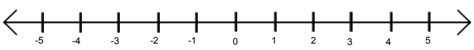
Example: $^-2 - {}^-4 = 2$



Number Line Model: You are standing on a number line facing the negative direction, then you move forward if adding a positive number or backward if adding a negative number.



$3 - 5 = {}^-2$



$^-3 - {}^-5 = 2$

Remark: Again, after thorough understanding is given, this following equation is useful to help students calculate subtraction quickly.

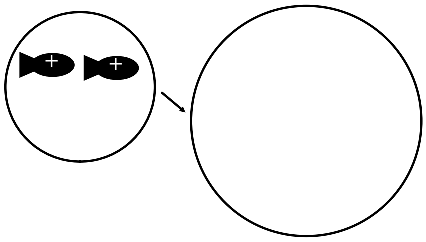
$$a - b = a + {}^-b$$

Subtraction of Integers:

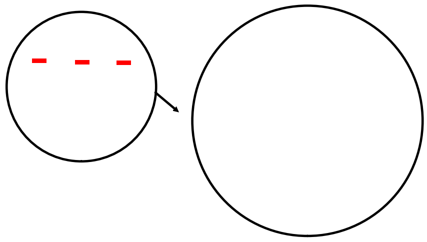
Definition: For integers a and b, $a - b$ is the unique integer n such that $a = b + n$.

Chip Model: Using the same setup as the chip model for addition, we want to take away the described numbers, and this may require us to create paired posi-fish and nega-fish.

Example: $2 - {}^-2 = 4$



Example: Draw the chip/charged field model for $^-3 - 1 = {}^-4$.



5.2 Notes

5.2: Multiplication and Division of Integers

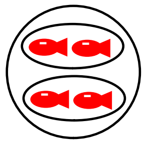
Multiplication of Integers:

Definition: For integers a and b,
(1) If $a > 0$, $a \times b = b + b + ... + b$ (a times).
(2) If $a = 0$, $a \times b = 0$.
(3) If $a < 0$, $a \times b = {}^-(|a| \times b)$.

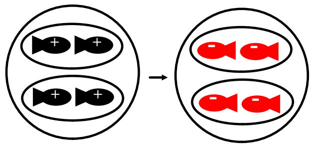
The order in this definition is more important than it may seem. The first value (ignoring negatives) is the number of groups, and the second number is what is in each group. If the first number is negative, we take the opposite after doing the multiplication.

Chip Model: Using the same setup as the chip model for addition, if $a > 0$, we represent a groups of b.

If $a < 0$, we represent $|a|$ groups of b, and then we convert to the other type of fish.



$2 \times {}^-2 = {}^-4$



${}^-2 \times 2 = {}^-4$

Charged Field Model: This model is the same as the chip model, but we use charges instead of fish.

Number Line Model: To compute $a \times b$:
1. Stand facing the direction of the sign of the first number.
2. Move the distance b forward if $b > 0$ or backward if $b < 0$.
3. Repeat this for a total of $|a|$ times.

$3 \times 2 = 6$

$3 \times {}^-2 = {}^-6$

${}^-3 \times {}^-2 = 6$

Example: Represent the problem $2 \times {}^-3 = {}^-6$ using each method.

Example: Represent the problem ${}^-2 \times {}^-3 = 6$ using each method.

Special Rules for Multiplication of Integers: The following set of rules can be taught to students who are struggling to speed up their calculations. This is NOT, however, a substitute for these models, as students should have a deep understanding of whether the solution is positive or negative before learning these.

1. If you are multiplying two numbers with the same sign, multiply their absolute values. (same signs positive)
2. If you are multiplying two numbers with different signs, multiply their absolute values and make this answer negative. (opposite signs negative)

5.2 Notes

Properties of Integer Multiplication:

Closure, Commutative, Associative, Identity still hold.

Distributive Property:

Zero Multiplication Property:

Other Properties:
7. $(\text{-}1)a = \text{-}a$

Division of Integers:

Definition: For integers a and b, $a \div b$ is the unique integer c, if it exists, such that $a = bc$.

Note: We say "if it exists" because $a \div b$ may not be an integer. We will not talk about a division algorithm with remainders, though it is the same with some slight restrictions on the choice of r.

Note 2: We will not learn any models for division, although a partition method with fish is a reasonable model.

Special Rules for Division of Integers: Again, this should not be taught until understanding takes place.

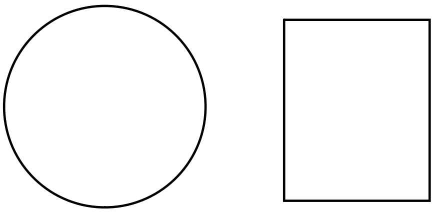
1. If you are dividing two numbers with the same sign, divide their absolute values. (same signs positive)
2. If you are dividing two numbers with different signs, divide their absolute values and make this answer negative. (opposite signs negative)

6.1 Notes

6.1: The Set of Rational Numbers

Definition: The rational numbers are all numbers of the form $\frac{a}{b}$, where a and b are integers with $b \neq 0$. We call a the numerator and b the denominator. We usually refer to these numbers in elementary school as fractions.

Example: Draw a figure to represent $\frac{1}{2}$ and $\frac{1}{3}$.



Definition: In the fraction $\frac{a}{b}$, if $|a| < |b|$, we call it a proper fraction. If $|a| \geq |b|$, we call it an improper fraction.

Example: List some proper and improper fractions.

Proper: Improper:

Question: Is every integer a rational number?

Fundamental Law of Fractions: If $\frac{a}{b}$ is any fraction and n is a nonzero integer, then $\frac{a}{b} = \frac{an}{bn}$.

Example: Show that $\frac{-7}{2} = \frac{7}{-2}$.

Example: Find a value for x such that $\frac{3}{12} = \frac{x}{72}$.

Example: Draw the points $\frac{-3}{2}$, 0, $\frac{3}{4}$, 2, $\frac{-7}{4}$, and $\frac{1}{2}$ on a number line.

Definition: Two fractions that represent the same rational number are known as equivalent fractions.

Example: Find fractions that are equivalent to $\frac{1}{2}$ by folding paper.

Definition: A rational number $\frac{a}{b}$ is said to be in simplest form if $b > 0$ and $\gcd(a,b) = 1$.

Example: Simplify the fraction $\frac{45}{300}$ by using the GCD.

6.1 Notes

Equality of Fractions: Show that $\frac{10}{16} = \frac{15}{24}$.

Theorem: Two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are equivalent if and only if $ad = bc$. That is, we can cross multiply to check these.

Proof:

Theorem: If a, b, and c are integers with $b > 0$, then $\frac{a}{b} > \frac{c}{b}$ if and only if $a > c$.

Example: Show that $\frac{9}{12} > \frac{6}{9}$.

Theorem: If a, b, c and d are integers with $b, d > 0$, then $\frac{a}{b} > \frac{c}{d}$ if and only if $ad > bc$.

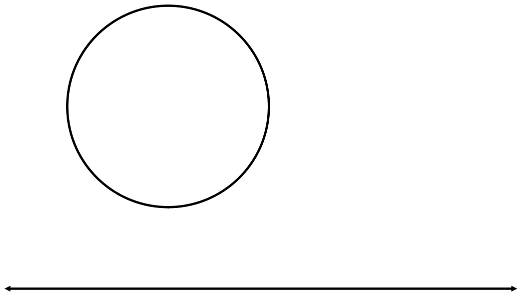
Proof:

6.2 Notes

6.2: Addition and Subtraction of Rational Numbers

Definition: If $\frac{a}{b}$ and $\frac{c}{b}$ are rational numbers, then $\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$.

Example: Draw a figure and a number line to represent $\frac{1}{4} + \frac{2}{4} = \frac{3}{4}$.



Theorem: If $\frac{a}{b}$ and $\frac{c}{d}$ are any two rational numbers, then $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

(Note: In practice, we usually find the least common denominator instead of just using this formula.)

Proof:

Definition: A mixed number is a number of the form $a\frac{b}{c}$, where a is an integer and $\frac{b}{c}$ is a proper fraction. The notation means $a\frac{b}{c} = a + \frac{b}{c}$.

A mixed number is a rational number, so we should be able to write it as $\frac{a}{b}$.

Example: Write the following numbers in the $\frac{a}{b}$ form.

(a) $2\frac{1}{4}$

(b) $-3\frac{2}{5}$

Example: Find a method of evaluating $\frac{1}{3} + \frac{1}{4}$.

Example: Calculate the following sums. Simplify your answers.

(a) $\frac{3}{10} + \frac{4}{15}$

(b) $\frac{3}{9} + \frac{2}{6}$

Example: Change the following fractions to mixed numbers.

(a) $\frac{22}{7}$

(b) $\frac{64}{19}$

6.2 Notes

Example: Calculate the following sums. Leave your answers as mixed numbers.

(a) $2\frac{3}{7} + 1\frac{11}{14}$

(b) $2\frac{1}{2} + 3\frac{2}{3}$

Which of our number properties are represented in the rational numbers over addition?

Closure:

Commutative:

Associative:

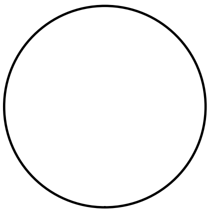
Identity:

Inverse:

Definition: If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then $\frac{a}{b} - \frac{c}{d}$ is the unique rational number $\frac{e}{f}$ such that $\frac{a}{b} = \frac{c}{d} + \frac{e}{f}$.

Note: An easy formula that we can get from this is $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$.

Example: Draw a figure to represent $\frac{5}{6} - \frac{2}{6} = \frac{3}{6}$.



Theorem: If $\frac{a}{b}$ and $\frac{c}{d}$ are any two rational numbers, then $\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$.

(Note: In practice, we usually find the least common denominator instead of just using this formula.)

Proof:

Example: Calculate the following. Simplify your answer.

(a) $\frac{3}{5} - \frac{1}{4}$

(b) $\frac{13}{16} - \frac{7}{30}$

Example: Calculate the following. Simplify your answer.

(a) $2\frac{1}{6} - 1\frac{9}{20}$

(b) $3\frac{1}{2} - 1\frac{5}{8}$

6.3 Notes

6.3: Multiplication and Division of Rational Numbers

Definition: If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers, then $\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$.

Example: Draw a figure to represent $\frac{1}{2} \cdot \frac{3}{4} = \frac{3}{8}$.

Example: Calculate $\frac{27}{62} \cdot \frac{8}{54}$.

Fact: The rational numbers over multiplication have the closure, commutative, and associative properites. The following properties also hold.

Identity:

Inverse:

Zero Multiplication Property:

Distributive:

Example: Draw a figure to represent $\frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}$.

Example: Calculate $\frac{18}{44} \cdot \frac{55}{27}$.

Example: Calculate the following.

(a) $3\frac{1}{3} \cdot 3\frac{1}{3}$

(b) $2\frac{2}{3} \cdot 1\frac{1}{4}$

Example: Show that $\frac{2}{3} \div \frac{3}{4} = \frac{8}{9}$.

Example: Compute $\frac{4}{5} \div \frac{12}{5}$ using Keep Change Flip with one of the explanations from before.

Definition: If $\frac{a}{b}$ and $\frac{c}{d}$ are rational numbers with $\frac{c}{d} \neq 0$, then $\frac{a}{b} \div \frac{c}{d}$ is the unique rational number $\frac{e}{f}$ such that $\frac{c}{d} \cdot \frac{e}{f} = \frac{a}{b}$.

We will not be studying a model for this in class, but look at p. 390 for some ideas of how to teach this.

Theorem: If $\frac{a}{b}$ and $\frac{c}{d}$ are any rational numbers and $\frac{c}{d} \neq 0$, then $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c}$.

Proof:

7.1 Notes

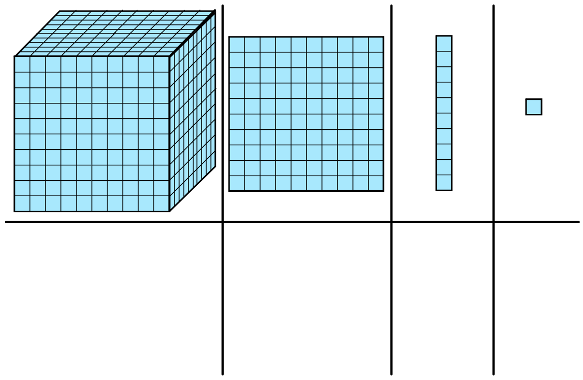
7.1: Introduction to Decimals

Definition: A decimal number is a notation to represent the sum of a whole number and fractions whose denominator is a power of 10.

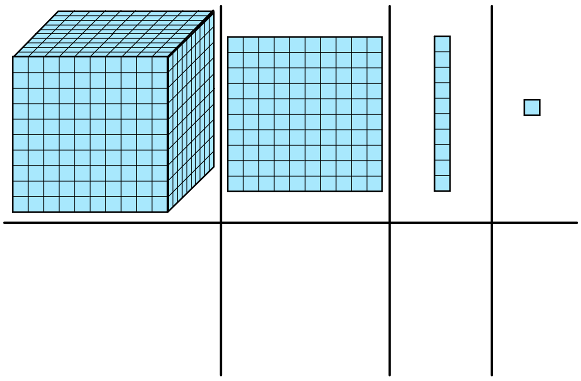
Example: Represent $1+\frac{3}{10}+\frac{6}{100}+\frac{2}{1000}$ as a decimal number.

Definition: The " . " above is called the decimal point.

Example: Write the number 1.236 as a sum of fractions, then represent it using base 10 blocks.



Example: Write the number 1.049 as a sum of fractions, then represent it using base 10 blocks.



Every place value is named after the denominator of its corresponding fraction.

Example: Circle the hundredths and ten-thousandths place in the following number.

3.14159

Question: Why does it not matter if we write additional zeroes at the end of a decimal number? (For example, 1.500 = 1.5)

A more standard interpretation is to represent the fraction as the whole number plus the entire decimal over a common denominator.

Example: Write 16.23 in this manner.

Example: Write 1.0495 in this manner.

7.1 Notes

A decimal number is read by saying the whole number, "and" the decimal part as a fraction.

Example: Write a out the way to read the following numbers.

(a) 16.23

(b) 1.0495

Example: Write each of the following as a fraction in simplest form.

(a) 0.625

(b) 1.42

(c) 0.1144

Example: Write each of the following as a decimal.

(a) $\frac{625}{10000}$

(b) $\frac{11}{125}$

(c) $\frac{27}{40}$

(d) $\frac{1}{32}$

Definition: A terminating decimal is a decimal that can be written with a finite number of digits after the decimal point.

Example: Try to write $\frac{1}{3}$ as a decimal number.

Theorem: A rational number $\frac{a}{b}$ in simplest form can be written as a terminating decimal if and only if the prime factorization of the denominator contains no primes other than 2 or 5.

Proof:

Example: Which of the following can be written as terminating decimals?

(a) $\frac{1}{8}$

(b) $\frac{8}{675}$

(c) $\frac{25}{98}$

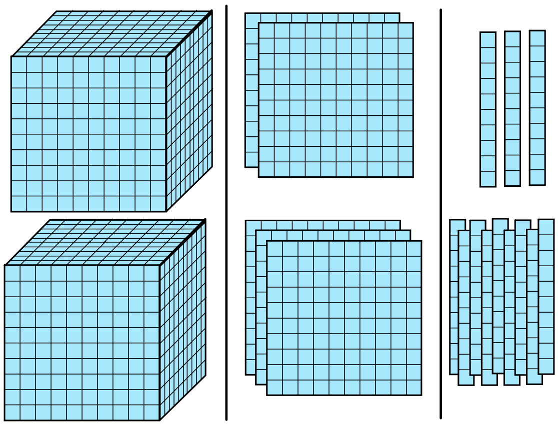
(d) $\frac{22}{265}$

7.2 Notes

7.2: Operations on Decimals

Example: (Addition of Decimals) Compute 1.23 + 1.39.

Example: Compute 1.23 + 1.39 by using Base 10 Blocks.



Example: Compute 1.2 + 2.104.

Example: Compute 1.52 - 1.334.

Summary: To add or subtract decimals:

1. Line up at decimal point.
2. Add zeroes in blank place values.
3. Standard Algorithm.
4. Put decimal point in same place as where it was lined up.

Why does this work?

Example: Compute 1.2×1.63.

7.2 Notes

To multiply decimals:

1. Use the standard algorithm, ignoring the decimals. Note: You do not need to align at the decimal point.
2. Move the decimal place to the left the same number of decimal places as the total of the number of decimal places in the two numbers.

Why does this work?

Example: Compute 1.53×0.74.

Example: Compute $132 \div 8$ as a decimal.

Summary: To divide to integers as a decimal, we divide as normal, but when we run out of numbers to bring down, we write a .0 (and a decimal point right above it) and bring down the 0. We then add additional zeroes as needed.

(Note: If the dividend is a decimal and the divisor is an integer, we put the decimal place above the dividend's decimal place and just add additional zeroes as needed.)

Why does this work?

Rounding Decimals: To round a decimal, we consider the number in the given place. If the digit after it is 5 or higher, we raise this number by 1 and remove the remaining digits. If the digit after it is 4 or less, we just remove the remaining digits.

Example: Round 1.3546 to the nearest thousandth.

Example: Round 1.922 to the nearest hundredth.

Example: Round 1.95 to the nearest tenth.

Example: Compute $22.57 \div 1.1$. Round to the nearest hundredth.

7.2 Notes

Summary: To divide by a decimal, move the decimal place of the divisor to the right until it is an integer, and then move the decimal place of the dividend the same number of places. Then divide as normal.

Why does this work?

Example: Compute 9.16 ÷ .365. Round to the nearest hundredth.

7.3 Notes

7.3: Nonterminating Decimals

Fact: The fraction $\frac{a}{b}$ is equivalent to the problem $a \div b$.

Example: Convert $\frac{3}{4}$ to a decimal.

Example: Try to write $\frac{1}{3}$ as a decimal number using division.

Definition: A repeating decimal is a decimal whose digits repeat every fixed number of digits.

Example: Convert the following fractions to decimals.

(a) $\frac{5}{11}$

(b) $\frac{2}{9}$

(c) $\frac{1}{7}$

Example: Convert the following repeating decimals to fractions.

(a) $0.\overline{3}$

(b) $0.\overline{09}$

(c) $0.1\overline{6}$

(d) $0.0\overline{45}$

7.3 Notes

(c) $0.\overline{27}$

(d) $0.\overline{126}$

Ordering Decimals:

To determine which of two decimals is larger, we consider their fraction representation.

Example: Show that 1.051 > 1.0495.

Example: Fill in the blanks with <, >, or =.

1.234 ___ 0.345

5.62 ___ 5.621

2.207 ___ 2.211

$1.\overline{45}$ ___ $1.45\overline{4}$

$1.\overline{36}$ ___ 1.363

(c) $0.2\overline{27}$

(f) $0.\overline{9}$

In summary, to order decimals, we consider the following:

1. If the whole number is larger, then the decimal is larger.
2. If the whole number is smaller, then the decimal is smaller.
3. If the whole numbers are equal, then we consider each decimal place (adding zeroes if necessary) until we reach the first decimal place where the digits are different. Whichever number has the larger digit is the larger number.
4. If the whole numbers and all the digits are the same, the decimals are equal.

Notice that everything we've done so far with decimals were converted from fractions. Are there decimals that can't be written as fractions?

Definition: The irrational numbers, denoted I, are the real numbers that are not rational numbers.

What would the irrational numbers look like as decimals?

6.4, 7.4 Notes

6.4, 7.4: Ratios, Proportions, and Percents

Definition: A ratio is a comparison between two given quantities and is usually written with a : or as a fraction.

Example: If there are 7 boys and 20 girls in a class, what is the

(a) boy to girl ratio?

(b) boy to class ratio?

(c) girl to class ratio?

Note: Ratios do not tell quantities but relative quantities.

Example: If just given a boy to class ratio of 1:2, we cannot determine how many boys are in the class.

Example: Given that the class size is 36 and the boy to class ratio is 1:3, determine the number of boys and girls in the class.

Definition: A proportion is a statement that two given ratios are equal.

Example: Show that the ratio of 1 cup of orange juice concentrate to 2 cups water (1:2) is equivalent to 3 cups concentrate to 6 cups water.

Fact: All rules of fractions apply to ratios, and all rules of equality of fractions apply to proportions.

Theorem: If a, b, c, and d are real number with $b \neq 0$ and $d \neq 0$, then the proportion $\frac{a}{b} = \frac{c}{d}$ is true if and only if $ad = bc$.

Proof: Consider this as equality of fractions, which was already proven.

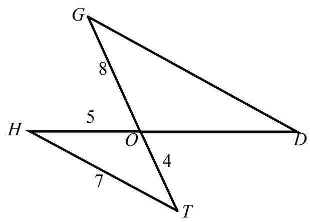
Example: If it takes 1 cup of flour to bake 24 cookies and you want to bake 60 cookies, how many cups of flour will you need?

Cross Multiplication is a nice method, but we should ensure that students gain understanding prior to teaching them this. Let's find some alternative methods in the following example.

Example: You go to the grocery store to buy a jar of peanut butter. The 16 oz jar costs $1.79 and the 40 oz jar costs $3.79. Which is a better deal?

Proportions come up in Math 222. Consider the following example.

Example: The two triangles below have the property that $\Delta HOT \sim \Delta DOG$. (Meaning corresponding sides have the same proportions.) Find the values of x and y.



6.4, 7.4 Notes

Definition: A percent, written with % at the end, is the number of parts out of 100 represented by a given number. That is $n\% = \frac{n}{100}$.

What is 1 whole as a percent?

What would 200% represent?

Example: Convert the following fractions to percents.

(a) $\frac{2}{5}$

(b) $\frac{3}{20}$

Example: Convert the following decimals to percents.

(a) 0.23

(b) 0.4785

(c) $1.1\overline{6}$

Let's come up with a rule for coverting decimals to percents.

Example: Convert the following percents to fractions.

(a) 18%

(b) $33.\overline{3}\%$

The following are types of problems your students may encounter.

What is 20% of 35?

14 is what percent of 35?

21 is 60% of what number?

At a local grocery store, 15% of the vegetables met the Organic classification. If the store has 2500 vegetables in stock, how many of the vegetables met the Organic classification?

The following are types of problems your students may encounter.

There are 320 frozen vegetables in a bag containing corn, green beans, and peas. If the bag contains 144 peas, what percentage of the vegetables in the bag are peas?

A laptop is bought and then sold one year later for $630, 10% less than what was originally paid for it. How much was paid for the computer originally?

Wal-Mart has a discount item rack that has an old version of a calculator on sale for 10% off, which amounts to a $5 discount. How much was the calculator originally? If the manager increases the discount to 20%, how much does the calculator cost now?